中国科学技术大学
University of Science and Technology of China

PPPL

A new coil design code FOCUS for designing stellarator coils without the winding surface

Caoxiang Zhu (祝曹祥)

University of Science and Technology of China, No. 96 JinZhai Road, Hefei, Anhui 230026, P. R. China

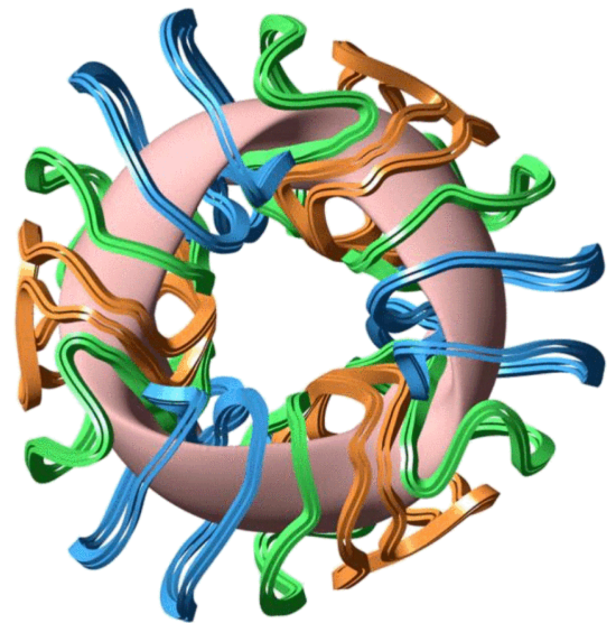
In collaboration with: Stuart Hudson, Samuel Lazerson, Nikolas Logan, Joshua Breslau, Neil Pomphrey, David Gates, Stewart Prager (PPPL); Aaron Bader, Thomas Kruger (UW-Madison); Yasuhiro Suzuki (NIFS), Yuntao Song (ASIPP); and Yuanxi Wan (USTC).

Introduction

> *"In the history of controlled thermonuclear fusion, there have been no ideas comparable in beauty and conceptual significance with that of the stellarator."*
>
> -- V. D. Shafranov, 1980 [*Helander, RPP, 2014*]

The difficulties in designing stellarator coils have been a critical problem for long time, even partly causing the termination of NCSX (*Neilson, et al., IEEE, 2010*) and the delay of W7-X construction (*Riße, FED, 2009*).

How to get the "best" coils for a given configuration?



Modular coils (non-planar) and the plasma for NCSX (*Neilson, et al., IEEE, 2010*)

Previous approaches require a defined winding surface.

❖ Surface current approximation

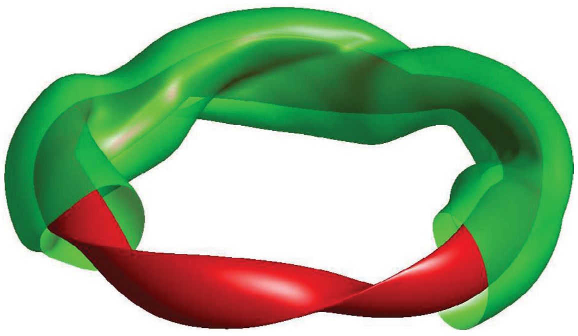
NESCOIL (*Merkel, NF, 1987*), **NESVD** (*Pomphrey, et al., NF, 2001*) and **REGCOIL** (*Landreman, NF, 2017*) assume that the magnetic field is produced by a surface current on a closed toroidal surface surrounding the plasma ("**winding surface**").

$$\mathbf{j} = \mathbf{n_c} \times \nabla \Phi$$

The goal is to minimize the normal fields on the prescribed plasma boundary

$$\epsilon^2 = \oint_p (\mathbf{B} \cdot \mathbf{n_p})^2 \, ds = 0$$

This is a least square fitting problem and Φ can be linearly solved. **Later discretized coil filaments are approximated along the surface current contours.**

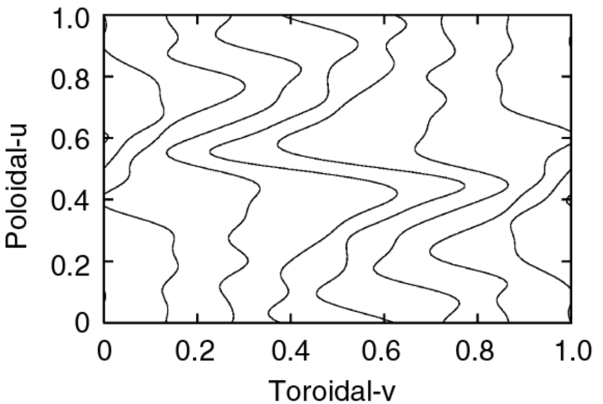


A winding surface (green) and the target plasma boundary (red) for W7-X (*Landreman, NF, 2017*)

❖ Direct nonlinear optimization

ONSET (*Drevlak, FST, 1998*), **COILOPT** (*Strickler, et al., FST, 2002*) and **COILOPT**++ (*Breslau, et al., EPR2013*) represent coils as 2D curves lying on a defined winding surface and apply nonlinear optimization algorithms to find coils that meet physics requirements and satisfy engineering constraints.

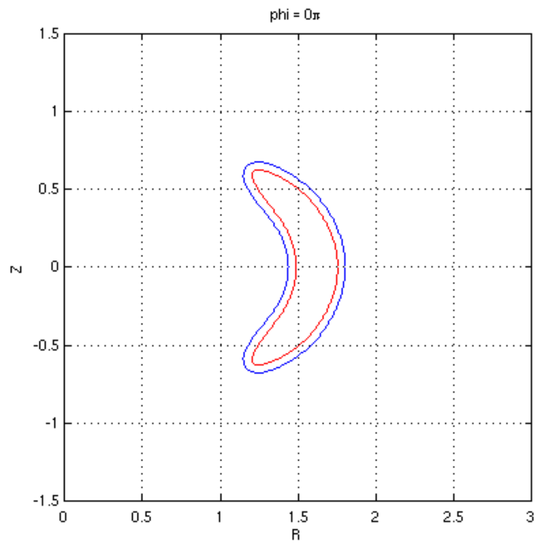
Typical engineering constraints include coil curvatures, coil torsions, coil length, coil-coil separation, coil-plasma separation, etc.



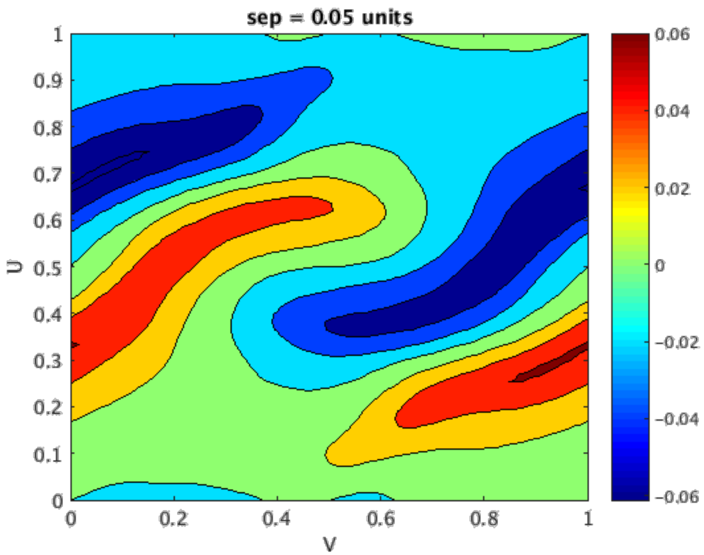
COILOPT optimized modular coils for NCSX.

Bad winding surface produce bad coils.

❖ All the methods need a pre-supposed winding surface. A “bad” winding surface directly results in the failure of finding an acceptable coils set.



Red is the target plasma boundary of NCSX and the blue is an winding surface produced by uniformly expanding the plasma surface with different distance.



Current potential distribution solved by NESCOIL on different winding surfaces. Final coils will be approximated from the contours.

❖ A “good” winding surface must be obtained first. But there are infinite choices. How can we get a good winding surface?

New method – NO winding surface

Flexible Optimized Coils Using Space-curves (FOCUS)

- ❑ Coil representations
- ❑ Physics constraints
- ❑ Engineering constraints
- ❑ Optimization algorithms
- ❑ Code structure

Fourier series are used to represent the 3D coils.

❖ How to represent an arbitrary closed curve in 3D space?

Fourier series:

$$\begin{cases} x(t) = X_{c,0} + \sum\limits_{n=1,N} [X_{c,n}\cos(nt) + X_{s,n}\sin(nt)] \\ y(t) = Y_{c,0} + \sum\limits_{n=1,N} [Y_{c,n}\cos(nt) + Y_{s,n}\sin(nt)] \\ z(t) = Z_{c,0} + \sum\limits_{n=1,N} [Z_{c,n}\cos(nt) + Z_{s,n}\sin(nt)] \end{cases}$$

Define $t \in [0,2\pi]$, an arbitrary parameter;

❖ A coil is fully determined by the its Fourier coefficients. All the free variables are

$$\mathbf{X} = \left[\overbrace{\underbrace{X^1_{c,0}, \cdots, X^1_{c,N}}_{\text{N+1}}, \underbrace{X^1_{s,1}, \cdots, X^1_{s,N}}_{\text{N}}, Y^1_{c,0}, \cdots, Z^1_{s,N}, I^1}^{\text{6N+4}}, \cdots, \cdots, X^{N_c}_{c,0}, \cdots, I^{N_c} \right]$$

❖ Fourier representation is general and differentiable.

* *The Fourier representation is not essential and can be replaced with others.*

Main physics constraints are normal fields and toroidal flux.

The external magnetic field within a region enclosed by a toroidal surface can be uniquely described by (1) the normal magnetic field on the torus; (2) the net toroidal magnetic flux within the torus.

❖ Match the normal field

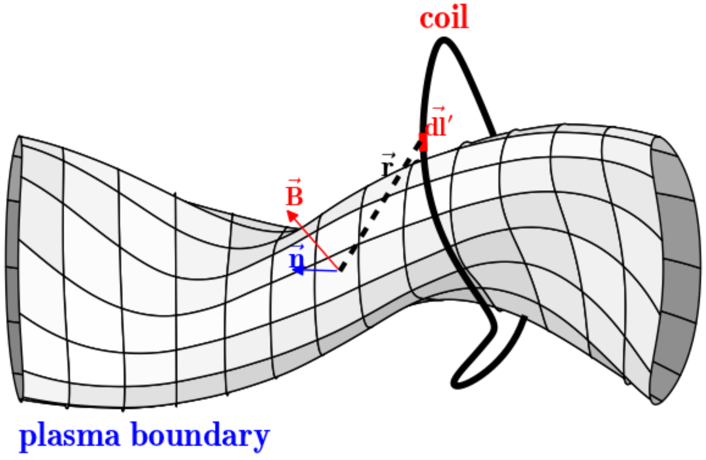
If the plasma boundary is a flux surface, the normal components of the total magnetic field would vanish. To achieve this, we can minimize the surface integral of the residue normal field.

$$f_B(\mathbf{X}) \equiv \int_S \frac{1}{2} (\mathbf{B} \cdot \mathbf{n})^2 \, ds$$

The total magnetic field comes from plasma current (from equilibrium codes) and the external coils (Biot-Savart Law)

$$\mathbf{B} = \mathbf{B}_P + \mathbf{B}_V$$

$$\mathbf{B}_V(\bar{\mathbf{x}}) = \frac{\mu_0}{4\pi} \sum_{i=1}^{N_C} I_i \int_{C_i} \frac{d\mathbf{l}_i \times \mathbf{r}}{r^3}$$



❖ Produce the flux

The toroidal magnetic flux is constant over the cross-sections of the plasma boundary.

$$f_\Psi(\mathbf{X}) \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \left(\frac{\Psi_\zeta - \Psi_o}{\Psi_o} \right)^2 d\zeta$$

Coil length penalty is the minimum engineering constraint.

In addition to physics requirements, engineering constraints should also be considered. The engineering constraints may or may not compete with the physics constraints.

❖ **The coil length constraint**

Without a constraint on the lengths of each coil, we found that the coils can become both arbitrarily long and can develop more "wiggles" so as to better produce the required magnetic field.

Including a penalty on the coil length could prevent this and it's also controlling the total materials used for building the coils.

$$f_L = \frac{1}{N_C} \sum_{i=1}^{N_C} \frac{e^{L_i}}{e^{L_{i,o}}} \qquad \textbf{OR} \qquad f_L = \frac{1}{N_C} \sum_{i=1}^{N_C} \frac{1}{2} \frac{(L_i - Li,o)^2}{L_{i,o}^2}$$

❖ **Chi-square minimization is used to optimize multiple objects.**

$$\min_{\mathbf{X} \in \mathbb{R}^n} \chi^2(\mathbf{X}) = \sum_i w_i \left(\frac{f_i(\mathbf{X}) - f_{i,o}}{f_{i,o}} \right)^2$$

$f_i(\boldsymbol{X})$ is the i-th function, with its objective value $f_{i,o}$ and a **user-specified** weight w_i.

❖ **The objective functions can be arbitrarily constructed. FOCUS has implemented minimum required constraints and retained the maximum flexibilities.**

The 1st and 2nd derivatives are calculated analytically.

Most nonlinear optimization algorithms require the gradient or even the Hessian. Usually, it's hard to differentiate the derivatives, especially for complex problems.

❖ **FOCUS can calculate the derivatives analytically.**

Example of calculating ∇f_B:

$$f_B(\mathbf{X}) \equiv \int_S \frac{1}{2} (\mathbf{B} \cdot \mathbf{n})^2 \, ds \qquad \frac{\partial f_B}{\partial X_i} = \int_S (\mathbf{B} \cdot \mathbf{n}) \left(\frac{\partial \mathbf{B_V}}{\partial X_i} \cdot \mathbf{n} \right) ds, \quad \forall X_i \in \mathbf{X}$$

$$\frac{\partial \mathbf{B}_V}{\partial X_i} = \frac{\mu_0}{4\pi} I \int_0^{2\pi} \frac{\delta \mathbf{B}_V}{\delta \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial X_i} dt \qquad \mathbf{B}_V(\bar{\mathbf{x}}) = \frac{\mu_0}{4\pi} \sum_{i=1}^{N_C} I_i \int_{C_i} \frac{d\mathbf{l}_i \times \mathbf{r}}{r^3}$$

$$\delta \mathbf{B}_V(\bar{\mathbf{x}}) = \frac{\mu_0}{4\pi} I \int_0^{2\pi} \left[\frac{3\mathbf{r} \cdot \mathbf{x}'}{r^5} \mathbf{r} \times \delta \mathbf{x} + \frac{2}{r^3} \delta \mathbf{x} \times \mathbf{x}' + \frac{3\mathbf{r} \cdot \delta \mathbf{x}}{r^5} \mathbf{x}' \times \mathbf{r} \right] dt$$

❖ **Analytically calculated derivatives are much faster and accurate.**

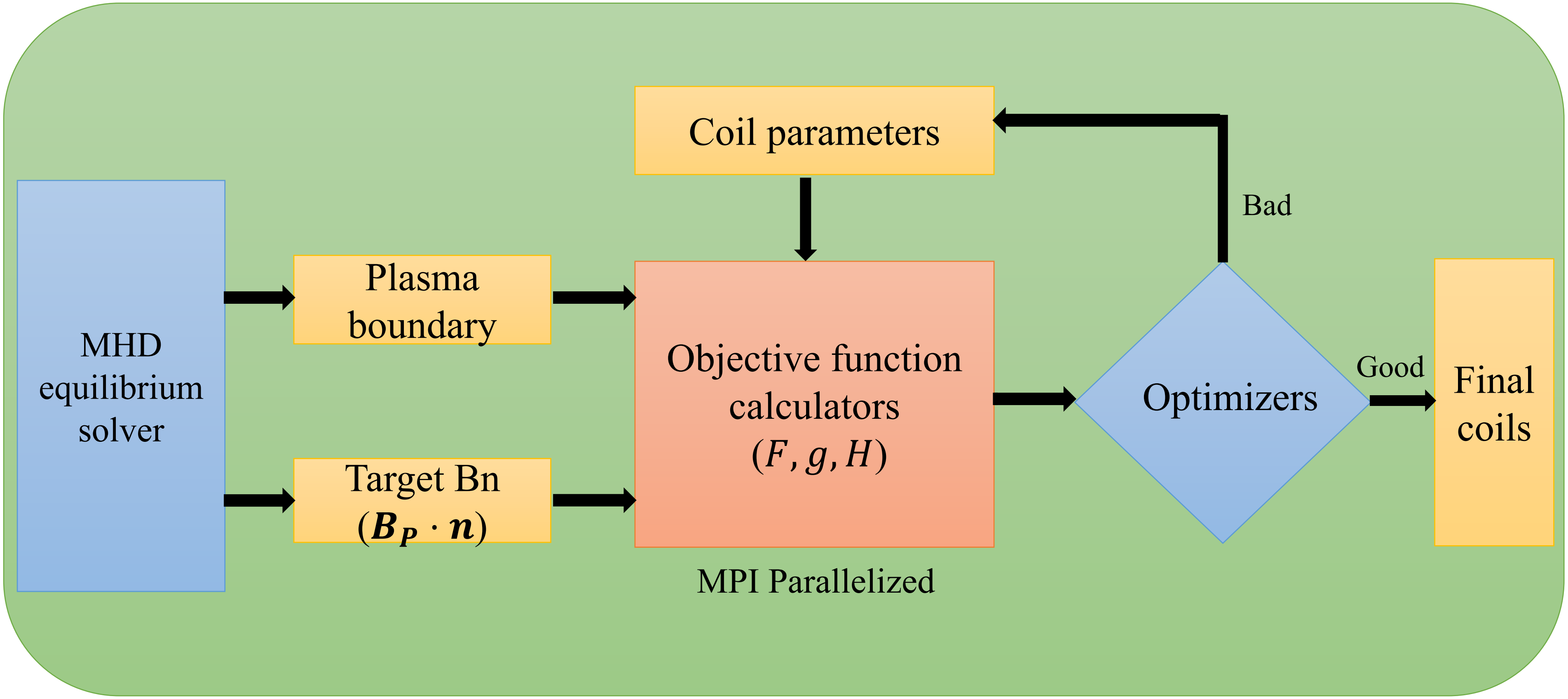
Speed for calculating the gradient: analytic (5s) VS central difference (574s)

❖ **Different nonlinear optimization algorithms are applied.**

- ✓ **Differential/Gradient Flow**
- ✓ **Nonlinear Conjugate Gradient**
- ✓ **Modified Newton Method**
- ✓ **Hybrid Powell Method**
- ✓ **Truncated Newton Method**

Basic structure of the code.

The basic optimization loop of FOCUS:



FOCUS has the flexibility to integrate with other stellarator optimization codes (even other coil design codes).

Applications

- **FOCUS has the ability to design various types of coils for different configurations.**

  - Modular coils for l=2 stellarator
  - Modular coils for W7-X
  - Helical coils for LHD

- **FOCUS can be used to analyze coil sensitivity on error fields.**

- **If more time,**

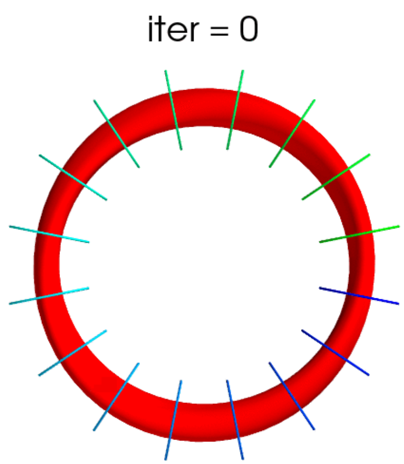
  - RMP coils for DIII-D
  - Improving coils designs for HSX

Design modular coils for a simple l=2 stellarator.

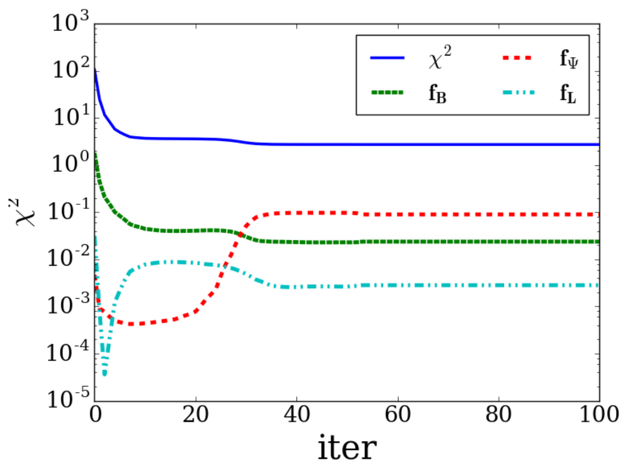
- **Target configuration:** a simple $N_P = 2$ rotating elliptical stellarator

$$R = 3.0 + 0.3\cos(\theta) - 0.06\cos(\theta - N_P\zeta),$$
$$Z = -0.3\sin(\theta) - 0.06\sin(-N_P\zeta) - 0.06\sin(\theta - N_P\zeta),$$

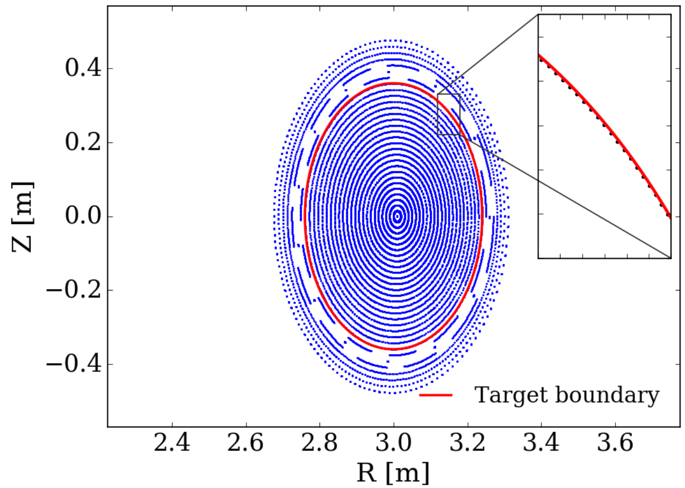
- **Initial coils guess:** 16 (8 per period) circular coils toroidally placed around the plasma.
- **Constraints**: f_B, f_Ψ, f_L
- **Optimizers**: Nonlinear conjugate gradient (50 iters) + Modified Newton method (50 iters)



Coil evolutions.



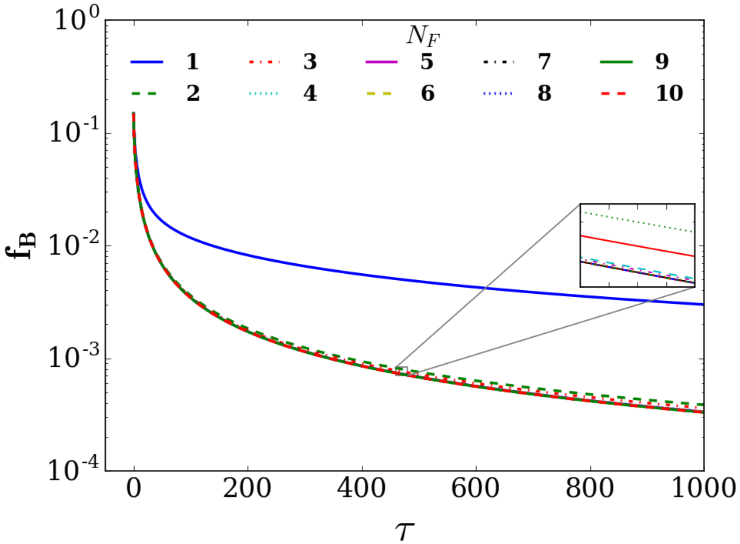
Descent of objective functions during the optimization.



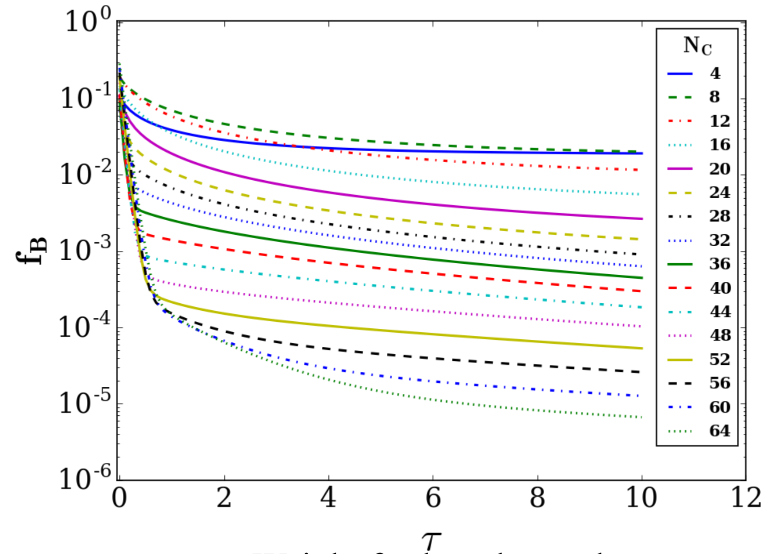
Realized flux surfaces compared to the target plasma boundary.

Convergence study with the rotating ellipse configuration.

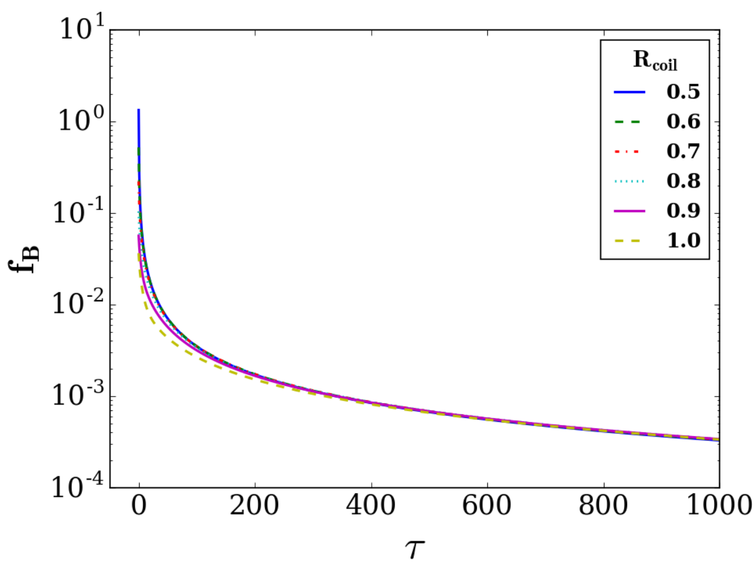
Number of Fourier harmonics for each coil



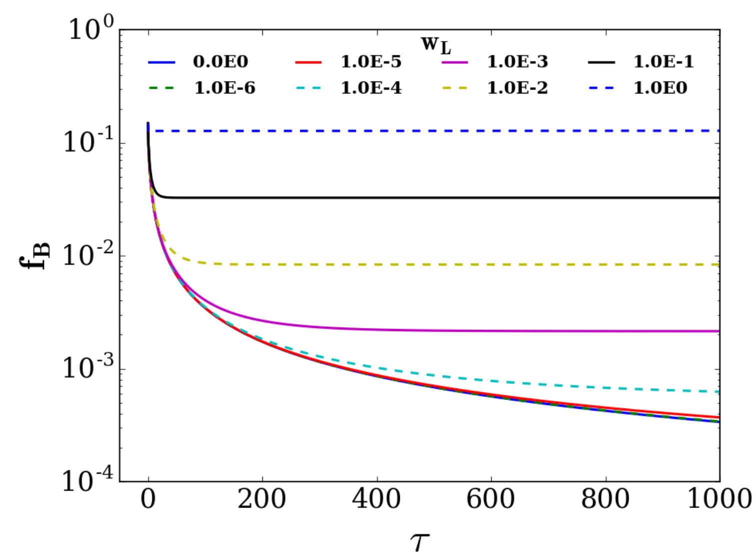
Different number of coils



Radius of initial circular coils

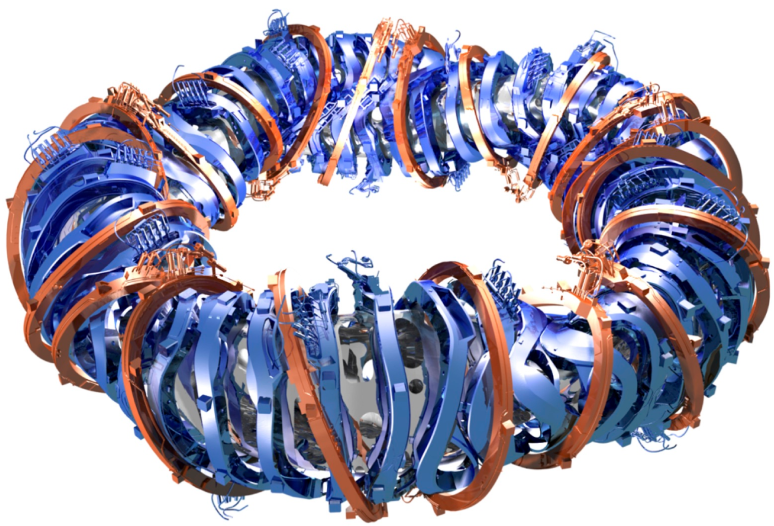


Weight for length penalty



W7-X stellarator and its coils system.

- Wendelstein 7-X (W7-X) is the largest stellarator in the world built in IPP, Germany.
- Basic parameters: $R_0 = 5.5$ m, $r = 0.53$ m, $N_{fp} = 5$, $B_{axis} = 3.0$T
- Built coils: 50 modular coils, 20 planar coils, some auxiliary coils



Overview of W7-X coil system including 50 modular coils (blue) and 20 planar coils (red). The planar coils are not used in the standard configuration.
From http://fusion.rma.ac.be/research.php?subj=W7X

How do the modular coils come from?

Optimization on the winding surface

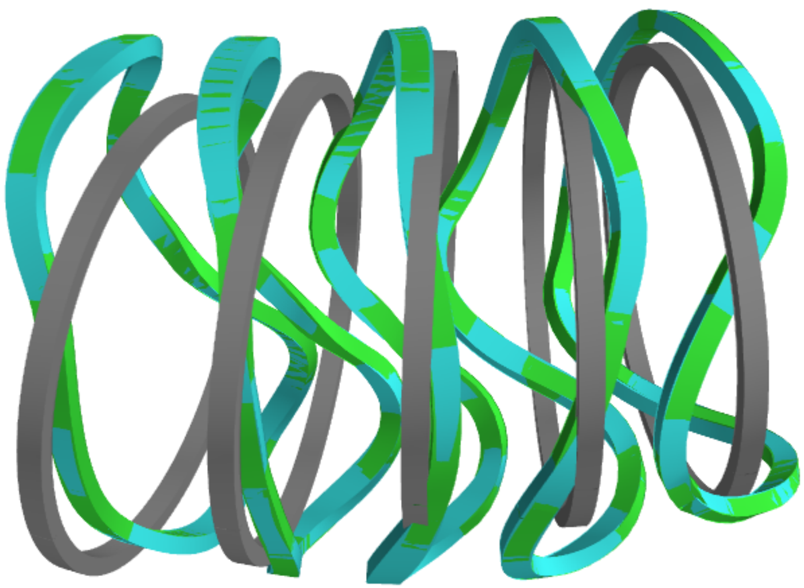
NESCOIL + **ONSET**

Reproducing W7-X coils from initial circular guesses.

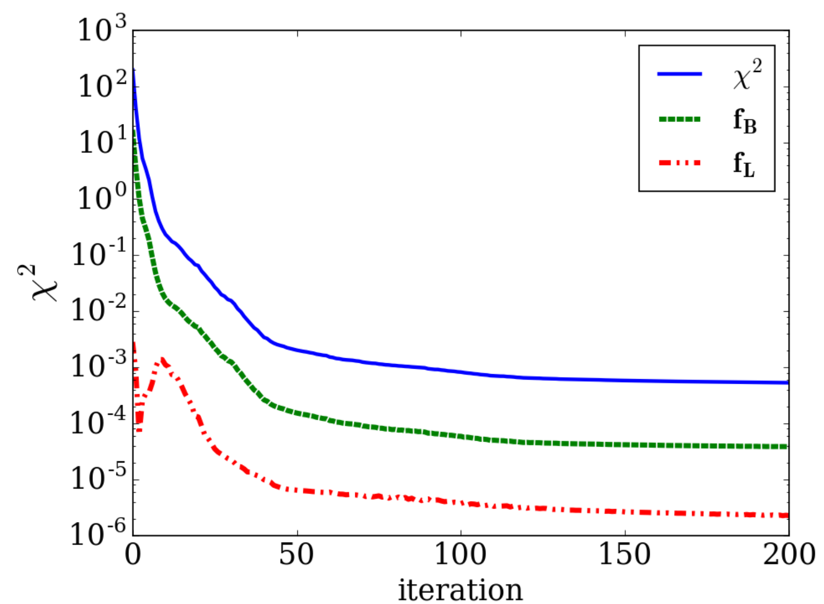
- **Target boundary:** an arbitrary toroidal surface with target Bn distribution;
- **Constraints:** minimize $\chi^2 = w_B f_B + w_L f_L$

$$f_B(\mathbf{X}) \equiv \int_S \frac{1}{2} \left(\mathbf{B_{coils}} \cdot \mathbf{n} - T_{Bn}\right)^2 ds. \qquad f_L(\mathbf{X}) = \frac{1}{N_C} \sum_{i=1}^{N_C} \frac{1}{2} \frac{(L_i - Li,o)^2}{L_{i,o}^2} \; .$$

- **Initial guesses**: 50 circular coils (r = 1.25m) equally placed surrounding the plasma;
- **Optimizer**: modified Newton method, 200 iterations, ~10h with 128 CPUs.



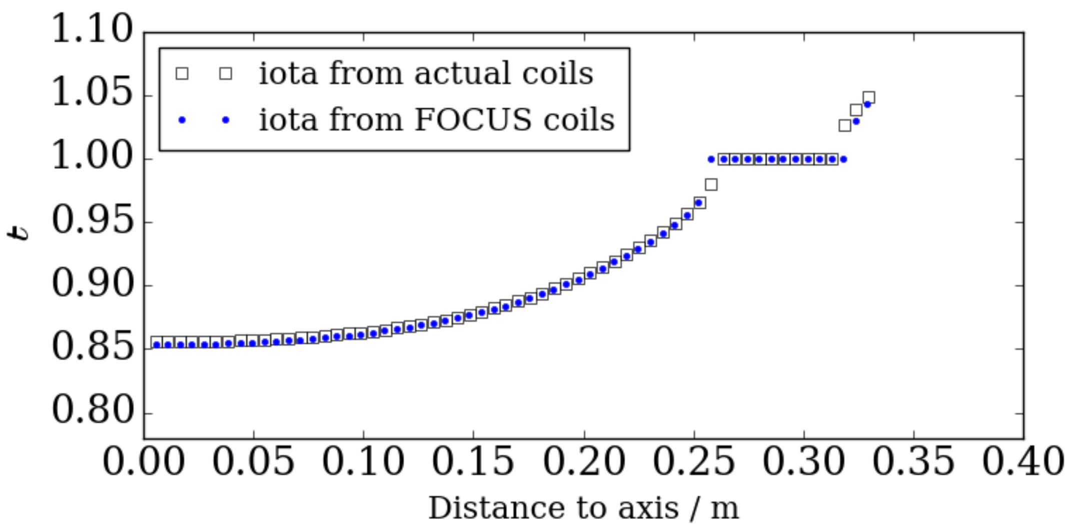
The initial circular coil (grey), the finally optimized coils (cyan) and the actual coils (green). Only five unique coils are plotted.



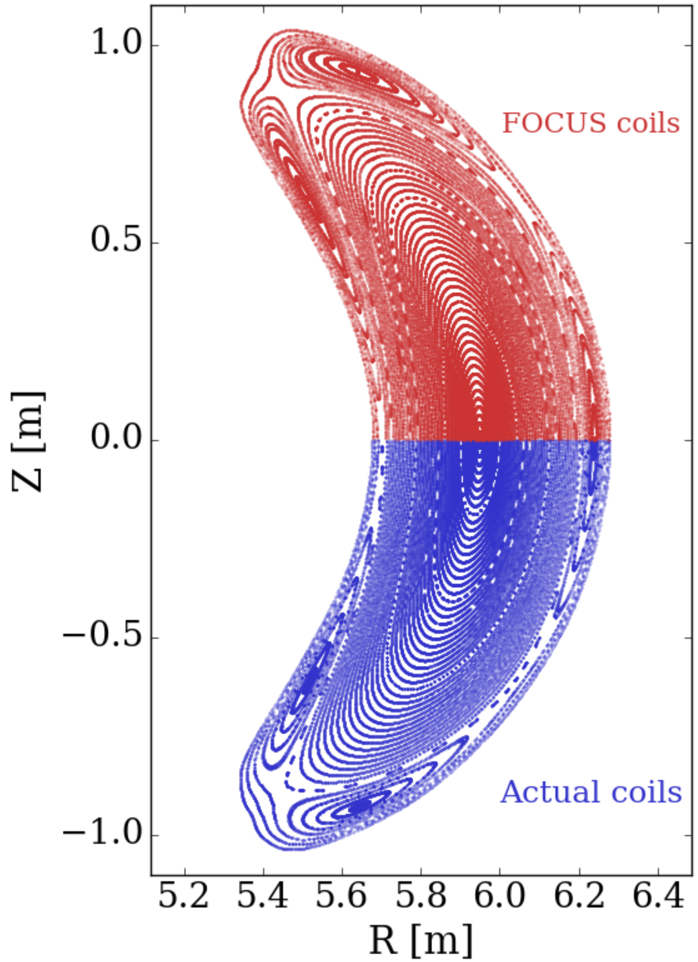
f_B decreases from 1.71×10^1 to 3.84×10^{-5} and f_L is reduced from 2.83×10^{-3} to 2.26×10^{-6}.

FOCUS coils produce the same iota profiles and Poincare plots.

Comparisons of the iota profiles and Poincare plots



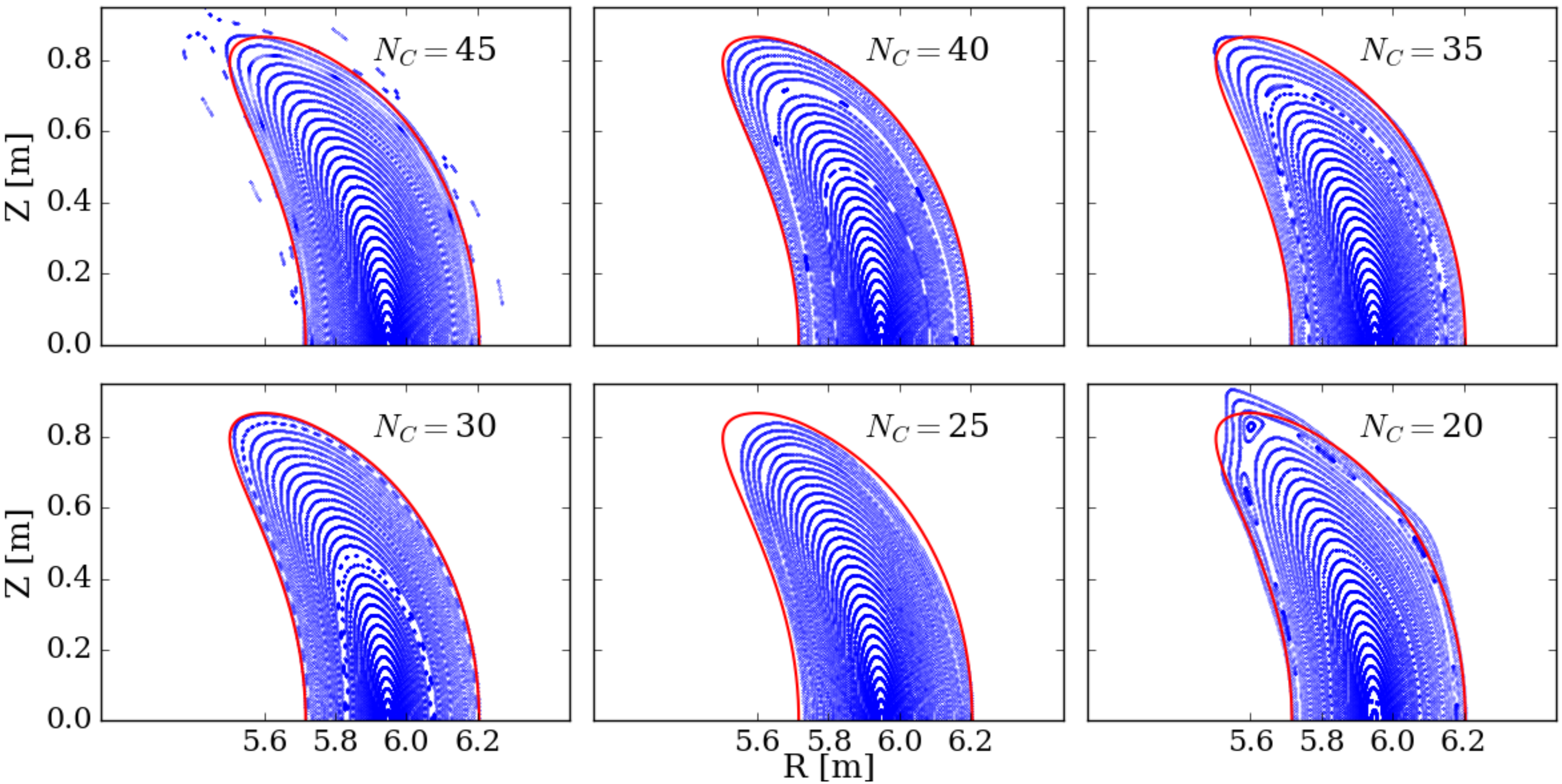
Comparison of the iota profiles. Abscissa is the distance to the magnetic axis (at the bean-shaped cross-section).



The produced Poincare plots of FOCUS coils (upper, red) and the actual coils (lower, blue).

Possible improvements: using as only as 30 coils?

Reduce the number of initial coils from 50 to 20 (red is the target boundary).



FOCUS can optimize LHD helical and VF coils.

- The LHD coil system can be approximated by two helical windings, together with three pairs of vertical field coils;
- None of the existing codes have been used to optimize helical coils; (previous work used analytical expressions)
- FOCUS can easily represent the helical and vertical field coils with Fourier representations;

l/m=2/10 helical windings:

$$x = R\cos(\theta) + \frac{1}{2} r\cos(4\theta) + \frac{1}{2} r\cos(6\theta)$$

$$y = R\sin(\theta) - \frac{1}{2} r\sin(4\theta) + \frac{1}{2} r\sin(6\theta)$$

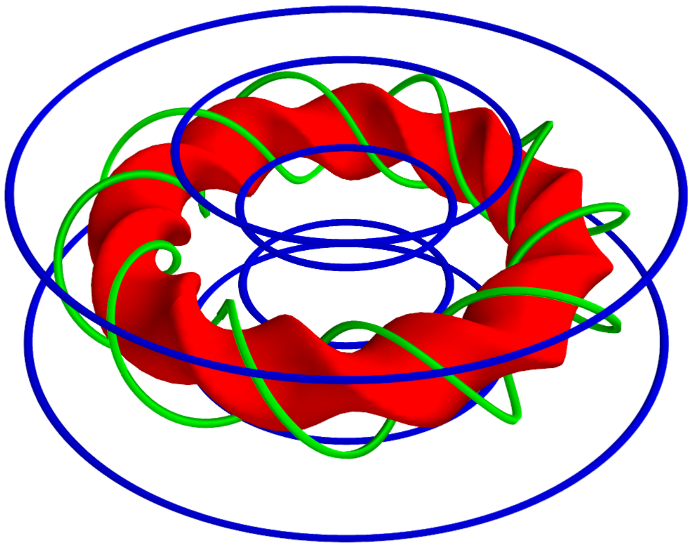
$$z = r\sin(5\theta)$$

Vertical filed coils:

$$x = r_0\cos(\theta)$$

$$y = r_0\sin(\theta)$$

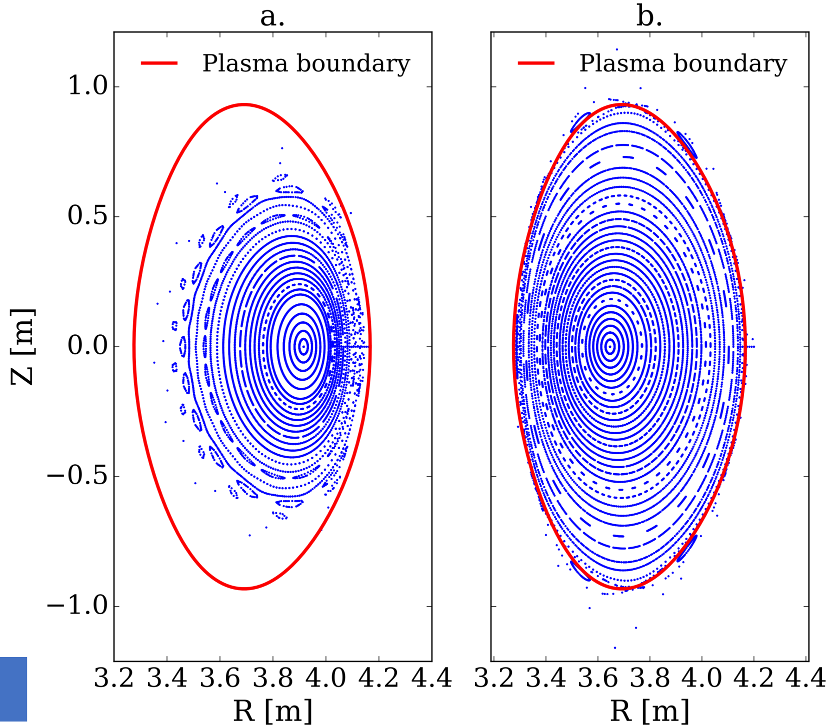
$$z = z_0$$



The LHD coils (simplified as filaments) including two helical windings (green) and six vertical coils (blue). (*data courtesy of Prof. Y. Suzuki from NIFS*)

Optimize helical coils:

1. Take the as-built coils data and fit with Fourier series;
2. Truncated errors are introduced and result in bad approximated magnetic field;
3. Use FOCUS to optimize the “bad” coils and coil parameters are adjusted to reproduce the target magnetic field.



Produced flux surface from fitted (a) and optimized (b) coils for LHD.

Vertical coils	IV r0/z0	IS r0/z0	OV r0/z0
fitted	1.8/0.8	2.82/2.0	5.5/1.55
optimized	1.764/0.7988	2.8061/1.9764	5.586/1.5302

Helical coil	X_c,1 / Y_s,1 (R)	X_c,4 / Y_s,4 (±1/2 r)	X_c,6 / Y_s,6 (1/2 r)	Z_s,5 (r)
fitted	3.85/3.85	0.49938/-0.49938	0.49938/0.49938	-0.99625
optimized	3.901/3.901	0.53296/-0.53296	-0.4574/-0.4574	-0.995

Coil sensitivity analysis on error fields using the Hessian method.

- ❖ Error fields are important to generate desired magnetic field for both tokamaks and stellarators.
- ❖ Not all the coil displacements have the same effect on the produced magnetic field.
- ❖ Coils sensitivity analysis can help isolate the most dominant displacement to avoid large error fields. And less attention should be paid to those insensitive displacements to save the cost and time.

Hessian method for sensitivity analysis.

If we have a multi-variables function $F(\boldsymbol{X})$, the Taylor expansion tells that

$$F(\mathbf{X}+\delta\mathbf{X}) = F(\mathbf{X}) + g^T\delta\mathbf{X} + \frac{1}{2}\delta\mathbf{X}^T H\delta\mathbf{X} + \mathcal{O}(\delta\mathbf{X}^3)$$

Suppose that $\boldsymbol{X}$ is **in the neighborhood of a local minimum** $\boldsymbol{X}^*$ and **$\delta\boldsymbol{X}$ is small**,

$$\delta F \approx \frac{1}{2}\delta\mathbf{X}^T H\delta\mathbf{X} = \frac{1}{2}\sum_i^n {a_i}^2\lambda_i$$

λ_i, $\boldsymbol{v}_i$ are the eigenvalues and eigenvectors of H. The perturbation is composed in eigen-space.

$$\delta\mathbf{X} = \sum_i^n a_i\mathbf{v}_i \qquad |\delta\mathbf{X}| = \sqrt{\sum_i^n {a_i}^2} = \xi\,, 0<\xi\ll 1$$

For perturbations with the same amplitude, the maximum change in F happens in the direction of eigenvector with largest eigenvalue.

Criteria for evaluating the error fields:

Normalized rms Bn

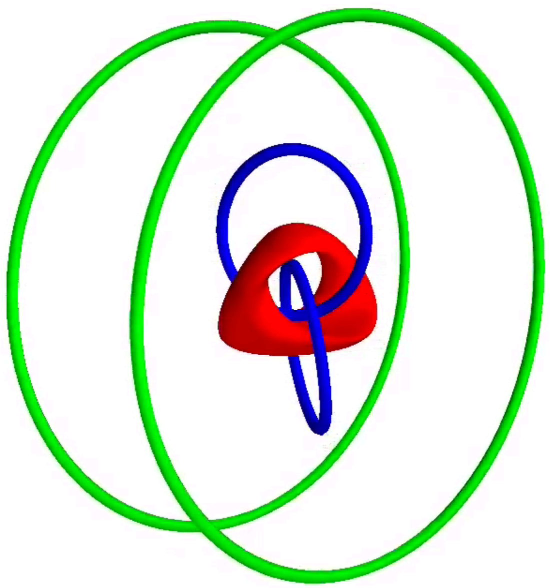
$$f_B(\mathbf{X}) \equiv \int_S \frac{1}{2}\left(\frac{\mathbf{B}\cdot\mathbf{n}}{|\mathbf{B}|}\right)^2 ds$$

Procedures of the Hessian method:

1) find a local minimum for f_B;
2) eigenvalue decomposing the Hessian matrix;
3) apply perturbations in the direction of different eigenvectors.

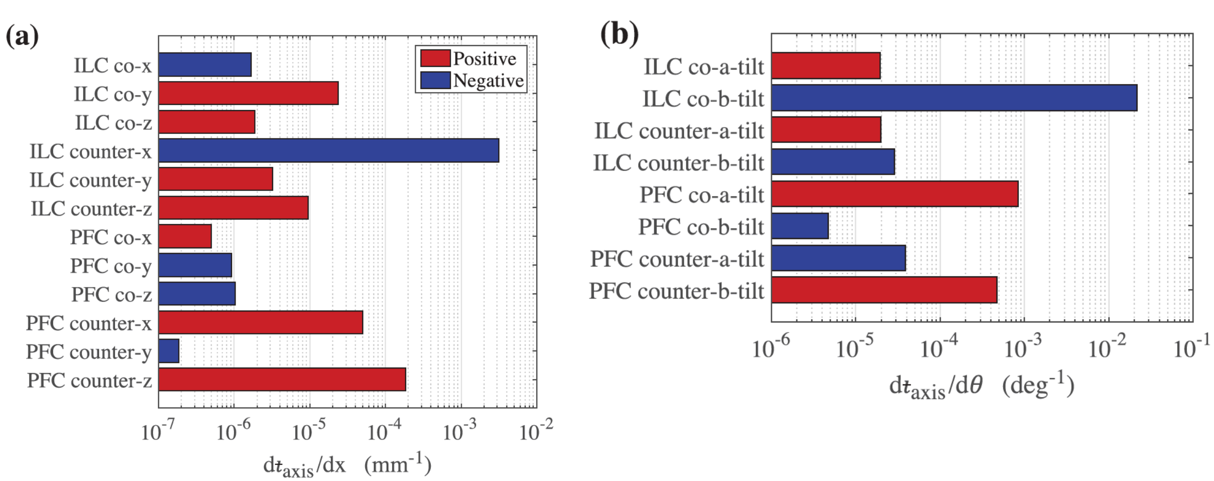
Previous study on error field analysis of CNT coils.

❖ CNT is a small stellarator in Columbia University for investigating non-neutral plasmas. It has two Helmholtz coils and two inter-linked (IL) circular coils.



CNT coils and plasma. *(data courtesy of S. Lazerson from PPPL)*

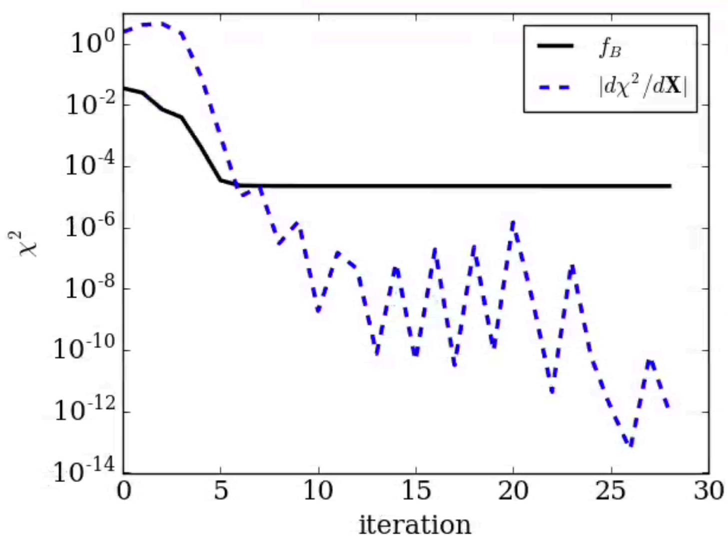
Hammond *et al.* carried out a numerical study showing that the greatest influences on ι_{axis} are **the separation of the IL coils** (along the z axis) and **the tilt angle between the IL coils** [*Hammond, et al., PPCF, 2016*].



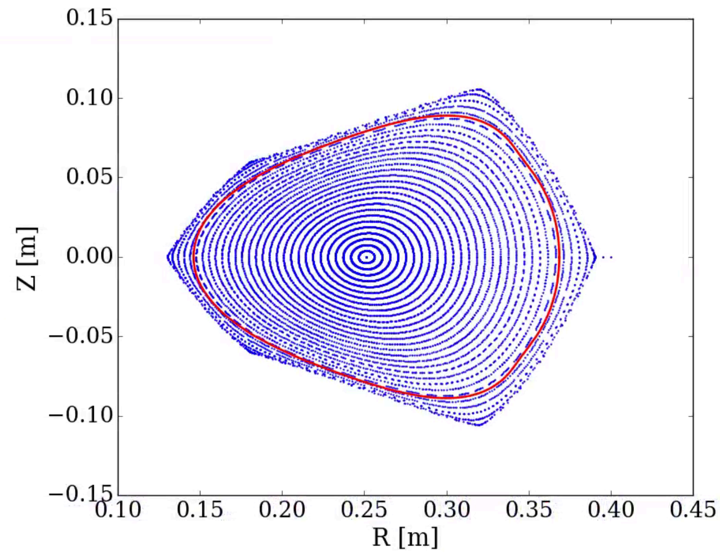
FOCUS find the optimal coils for CNT.

Find the minimum

- starting from the real coils but IL have zero tilt angle;
- N_F=1 to enforce planar coils;
- use the Truncated Newton Method to minimize f_B;



Descent of f_B and |g|.

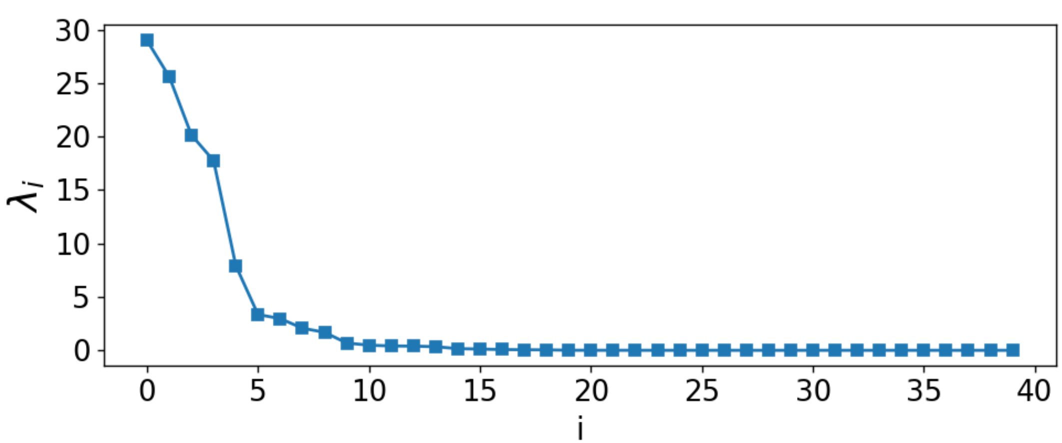


Realized vacuum flux surfaces.

Eigenvalue decomposing the Hessian

The Hessian is a 40×40 symmetric matrix. At the minimum, it's semi-positive definite (since only f_B).

The eigenvalues decay rapidly and only several large ones should be taken into account. The maximum eigenvalue is about 28.99, with the second of 25.61.

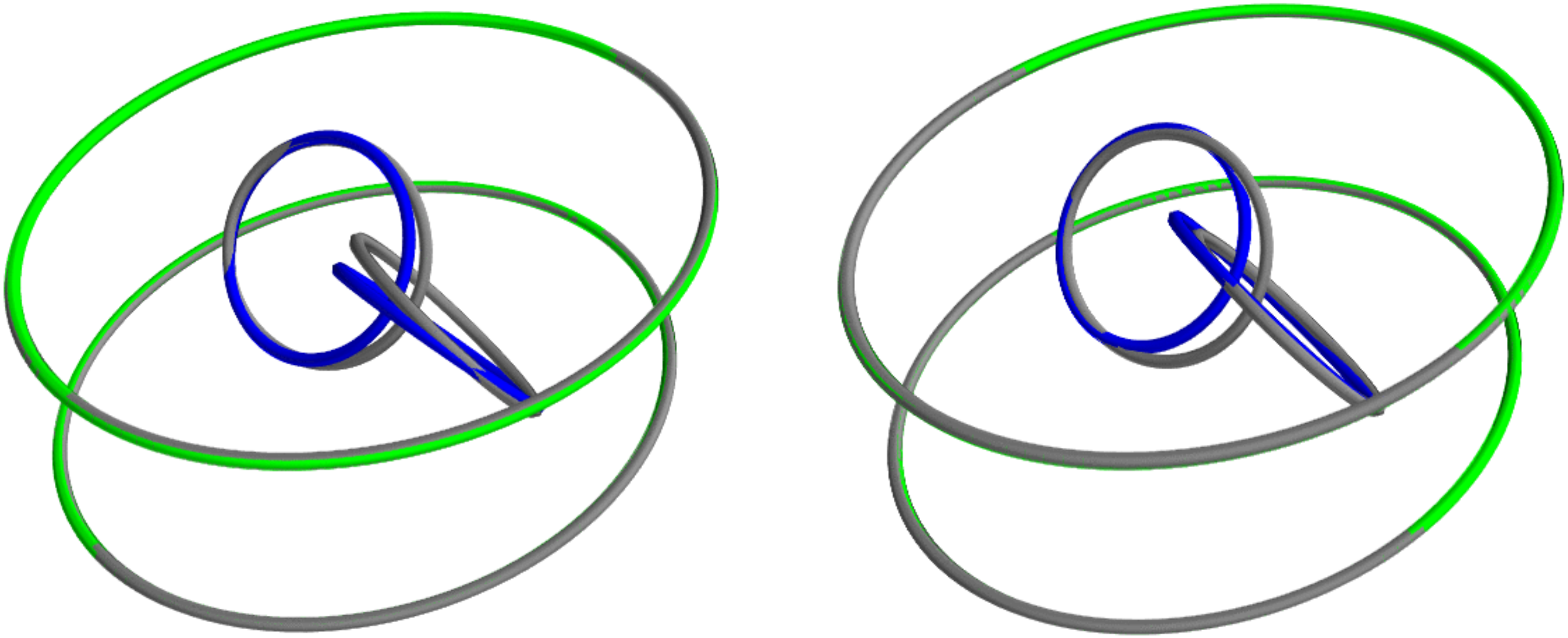


Eigenvalues spectra of the Hessian matrix.

Results from FOCUS are the same with Hammond's.

Apply two perturbations in the direction of eigenvector with the two largest eigenvalues.

$\boldsymbol{X}^{*} + \xi \boldsymbol{v}_{\mathbf{1}}$ $\boldsymbol{X}^{*} + \xi \boldsymbol{v}_{\mathbf{2}}$



Changes in the coil geometries under different magnitude ($-0.5 \leq \xi \leq 0.5$) of perturbations in the direction of eigenvector with the largest (left) and the second largest (right) eigenvalues. (grey stands for the "equilibrium" coils).

The perturbation in the left is actually **changing the tilt angle** and the right is **changing the separation of IL coils**. **The same results as Hammond's.**

Summary

❖ **Introduce a new method for designing stellarator coils, with two new features:**

- more flexible by getting rid of the "winding surface"
- more robust and faster by employing analytically calculated derivatives

❖ **Present several examples of using the new method to optimize various types of coils in different configurations, including**

1) modular coils for stellarators
2) helical coils for heliotrons
3) planar coils for CNT

The fact that with only targeting the length we can recover the W7-X real coils is exciting.

❖ **Demonstrate an elegant way to analyze coil sensitivities using the Hessian method.**

CNT results coincide with previous work by using numerical calculations.